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Physical Unitarity for Massive Non-abelian Gauge Theories in the Landau Gauge: Stückelberg & Higgs

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Abstract

We discuss the problem of unitarity for Yang-Mills theory in the Landau gauge with a mass term *à la* Stückelberg. We assume that the theory (non-renormalizable) makes sense in some subtraction scheme (in particular the Slavnov-Taylor identities should be respected!) and we devote the paper to the study of the space of the unphysical modes. We find that the theory is unitary only under the hypothesis that the 1-PI two-point function of the vector mesons has no poles (at $p^2 = 0$). This normalization condition might be rather crucial in the very definition of the theory. With all these provisos the theory is unitary. The proof of unitarity is given both in a form that allows a direct transcription in terms of Feynman amplitudes (cutting rules) and in the operatorial form.

The same arguments and conclusions apply *verbatim* to the case of non-abelian gauge theories where the mass of the vector meson is generated via Higgs mechanism. To the best of our knowledge, there is no mention in the literature on the necessary condition implied by physical unitarity.

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1 Introduction

The quest for a consistent non-abelian gauge theory [1] of massive gauge bosons is a subject with a long and venerable history. Today the preferred solution, combining unitarity and renormalizability, is still the spontaneous symmetry breaking mechanism based on the introduction of the Higgs field [2]. Nevertheless, within the context of non-power-counting renormalizable models, the Stückelberg mechanism [3] has been repeatedly advocated [4, 5] as a possible alternative for the generation of massive non-abelian vector fields.

This paper is devoted to the discussion of some crucial points in the proof of Physical Unitarity for a massive non-abelian gauge theory in the presence of a mass term *à la* Stückelberg. Such term has been originally introduced in order to have a gauge invariant theory for massive photons. It can be seen as the result of an operatorial gauge transformation on the fields in the Proca gauge. The same procedure can be envisaged also in the case of non-abelian gauge theories [6]-[11]. While in the abelian case the theory is renormalizable and moreover the proof of unitarity on the physical states poses no problems, the non-abelian case is far more complicated. The origin of the troubles is mainly the term generated in the mass by the operatorial gauge transformation: it yields a non-polynomial lagrangian. The efforts made in order to overcome these difficulties is a long list in the history of quantum field theory. Along this line one of the first steps to be accomplished is a close analysis of the problem of unitarity. In fact even if the theory is made finite by some subtraction scheme, physical unitarity will always be one crucial item to be considered. The present work is aimed at focusing on the conditions that have to be met in order to guarantee this important property. Previous attempts to prove unitarity made use of a direct diagrammatic study of the Feynman amplitudes and they are limited to one-loop in the perturbative expansion [11].

The classical lagrangian for $SU(2)$ is

$$\begin{aligned}\mathcal{L} &= -\frac{1}{4}G_{a\mu\nu}G_a^{\mu\nu} + m^2\ Tr\ \left(A_\mu + \frac{i}{g}\Omega\partial_\mu\Omega^\dagger\right)^2 + \mathcal{L}_M,\\ G_{a\mu\nu} &= \partial_\mu A_{a\nu} - \partial_\nu A_{a\mu} + \epsilon_{abc}A_{b\mu}A_{c\nu},\end{aligned} \tag{1}$$

where m is the Stückelberg mass and Ω is parameterized in terms of the

Stückelberg fields $\vec{\phi}$ by

$$m\Omega = \phi_0 \mathbf{1} + i\vec{\phi}\cdot\vec{\tau} \tag{2}$$

with $\phi_0 = \sqrt{m^2 - \phi_a^2}$.

The discussion is devoted to the particular formulation provided by the Landau gauge-fixing term

$$S_{g.f.} = \int d^4x \; 2\,Tr\left(B\partial^\mu A_\mu - \bar{c}\partial^\mu D[A]_\mu c\right). \tag{3}$$

We chose this gauge because with a transverse vector field propagator the Feynman rules are particularly simple. In particular there are no important out-of-diagonal terms in the connected two-point functions.

In Appendix A we elaborate upon other covariant gauges and demonstrate that the subspace of the unphysical modes includes also dipole fields, as it is known in power-counting renormalizable theories.

The presence of a further scalar mode, introduced via the Stückelberg mass term, requires a revisitation of the standard proof of physical unitarity in non-abelian gauge theories [12], [13], [14], [15]. In particular a detailed study of the Fock space is necessary in order to identify the unphysical modes. The usual method based on the study of the kernel of the BRST charge [14] [15]

$$|Phys\rangle \in \mathrm{Ker}\; Q/\mathrm{Im}\; Q \tag{4}$$

has to be supported by a preliminary study of the Fock space of the theory. In particular it appears that too many fields describe asymptotically massless unphysical modes (the vector field, the Nakanishi-Lautrup field [17][18] and the Stückelberg field). A condition has to be met in order that the definition of physical space in eq. (4) guarantees physical unitarity. The Landau gauge requires that the connected two-point function for the gauge bosons is transverse

$$W^{\mu\nu}_{ab} = \delta_{ab} W_T(p^2)\left(g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2}\right). \tag{5}$$

The main result of the paper is the following. If one can make sense out of a non-abelian gauge theory with a Stückelberg mass term, then the physical unitarity is satisfied provided one can impose the normalization condition

$$W_T(0) = \lim_{p^2=0} \frac{W_{\phi\phi}}{p^2 W^2_{B\phi}}, \tag{6}$$

where ϕ is the Stückelberg field. The importance of the result is quite transparent: if the condition cannot be enforced, unitarity (for the physical states) is lost. Thus this seems to be the very crucial condition to meet in order to define the theory.

The same discussion and the same condition (6) is valid for a massive non-abelian gauge theory where the mass is generated by the Higgs mechanism. In this case the rôle of the Stückelberg field is played by the unphysical components of the Higgs field. The proof here provided for the Stückelberg case is valid also for the Higgs case since the ingredients used are essentially the same: i) the Slavnov-Taylor identities [19], ii) the equation of motion of the Nakanishi-Lautrup field and iii) the equation of motion of the ghost field [20]. After this remark we have chosen to discuss only the Stückelberg case since we optimistically hope that some day the obstacle of non-renormalizability [21] will be removed and a theory, consistent from the point view of physics, will be thus achieved.

The paper is organized as follows. The abelian Stückelberg model is reviewed in Sect. 2. The non-abelian case is taken up in Sect. 3. The diagrammatic approach [22] to the analysis of physical unitarity is presented in Sect. 4, while the complementary operatorial approach [14, 15, 16] is considered in Sect. 5. Finally conclusions are given in Sect. 6.

2 Abelian case

As a warm-up we consider the Landau gauge in the abelian case. The Lagrangian is

$$\mathcal{L} = -\frac{1}{4}F^2_{\mu\nu} + \frac{m^2}{2}(A_\mu - \frac{1}{m}\partial_\mu\phi)^2 + B\partial_\mu A^\mu + \mathcal{L}_M\,, \tag{7}$$

where matter enters in $\mathcal{L}_M$. For abelian theories the Stückelberg mechanism leads to a power-counting renormalizable model.

The Ward identity is (dots indicate the part relevant for matter)

$$\Box W_B + \partial_\mu J^\mu_A - mJ_\phi + \ldots = 0 \tag{8}$$

the B equation of motion[3]

$$J_B + \partial^\mu W_{A^\mu} = 0\,, \tag{9}$$

[3]The notation is as follows: W_ψ stands for $\delta W/\delta J_\psi$, with ψ any of the quantized fields of the model and J_ψ its source. The connected generating functional $W[J_\psi]$ is related to the vertex functional $\Gamma[\psi]$ by $W = \Gamma + \int d^4x\, J_\psi\psi$.

and the ϕ equation of motion

$$J_\phi + m(\partial^\mu W_{A^\mu} - \frac{1}{m}\Box W_\phi) = 0. \tag{10}$$

It is useful to have these equations for the vertex functional

$$\Box B - \partial_\mu \Gamma_{A_\mu} + m\Gamma_\phi + \ldots = 0 \tag{11}$$
$$\Gamma_B = \partial^\mu A_\mu \tag{12}$$
$$\Gamma_\phi = m(\partial_\mu A^\mu - \frac{1}{m}\Box\phi). \tag{13}$$

One obtains easily

$$\begin{aligned}&W_{A^\mu\phi} = 0, \quad W_{\phi\phi} = -\frac{1}{p^2}, \quad W_{\phi B} = \frac{m}{p^2},\\&W_{BB} = 0, \quad W_{A^\mu(p)B} = -i\frac{p_\mu}{p^2}.\end{aligned} \tag{14}$$

The propagator of the gauge bosons is transverse

$$W_{A_\mu A_\nu} = \frac{1}{p^2 - m^2} T_{\mu\nu}\,. \tag{15}$$

where $T_{\mu\nu}$ is the projector

$$T_{\mu\nu} = g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\,. \tag{16}$$

We will also need later on the orthogonal longitudinal projector $L_{\mu\nu} = \frac{p_\mu p_\nu}{p^2}$. The final goal of the calculation is the construction of three linearly independent fields, spanning the bosonic sector at $p^2 = 0$, such that at the pole in $p^2 = 0$ one has

$$\begin{aligned}&W_{BB} = 0, \quad W_{BX} = \frac{1}{p^2}, \quad W_{XX} = 0,\\&W_{Y^\mu A^\nu} = 0, \quad W_{Y^\mu B} = 0, \quad W_{Y^\mu X} = 0.\end{aligned} \tag{17}$$

By using eqs. (14) one gets

$$X = \frac{1}{2m^2}B + \frac{1}{m}\phi. \tag{18}$$

Now we construct the field Y^μ

$$Y^\mu = aA^\mu + b\partial^\mu B + c\partial^\mu X \tag{19}$$

and we get

$$W_{Y^\mu A^\nu} = aW_{A^\mu A^\nu} + b\frac{p_\mu p_\nu}{p^2} + c\frac{p_\mu p_\nu}{2m^2 p^2} = 0 \tag{20}$$

$$W_{Y^\mu B} = ia\frac{p_\mu}{p^2} + ic\frac{p_\mu}{p^2} = 0 \tag{21}$$

$$W_{Y^\mu X} = ia\frac{p_\mu}{2m^2 p^2} + ib\frac{p_\mu}{p^2} = 0\,, \tag{22}$$

which admits a non-trivial solution only if the pole part of $W_{A_\mu A_\nu}$ at $p^2 = 0$ is

$$W_{A^\mu A^\nu} = \frac{p_\mu p_\nu}{m^2 p^2} \qquad \text{for} \quad p^2 \sim 0 \tag{23}$$

and then

$$\begin{aligned} Y^\mu &= a\left(A^\mu - \frac{1}{2m^2}\partial^\mu B - \partial^\mu X\right) \\ &= a\left(A^\mu - \frac{1}{m^2}\partial^\mu B - \frac{1}{m}\partial^\mu \phi\right). \end{aligned} \tag{24}$$

The condition in eq. (23) is related to a further property of the theory. Consider the vertex functional Γ. For the vector field we have

$$\Gamma_{\mu\nu} = \Gamma_T\Big(g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\Big) + \Gamma_L\frac{p_\mu p_\nu}{p^2}. \tag{25}$$

From the definition the two-point functions obey the equation

$$W\Gamma = -1. \tag{26}$$

Then we get

$$W_T\Gamma_T = -1. \tag{27}$$

and

$$W_{BA^\mu}\Gamma_{A_\mu A_\nu} + W_{B\phi}\Gamma_{\phi A_\nu} = 0. \tag{28}$$

By using eqs.(13) and (14)

$$i\frac{p_\mu}{p^2}\Gamma_{A_\mu A_\nu} + \frac{1}{p^2}(-im^2 p^\nu) = 0 \tag{29}$$

i.e.

$$\Gamma_L = m^2 \tag{30}$$

and finally

$$\Gamma_{\mu\nu} = -\frac{1}{W_T}\Big(g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\Big) + m^2\frac{p_\mu p_\nu}{p^2}. \tag{31}$$

The absence of singularities in Γ at $p^2 = 0$ requires

$$W_T(0) = -\frac{1}{m^2} \tag{32}$$

i.e. the condition in eq. (23). This condition becomes a non-trivial normalization condition if one introduces matter in the theory. The radiative corrections to the vector field propagator can be described by the 1-PI vertex function

$$\Pi^{\mu\nu}(p) = \Pi(p^2, M)(p^2 g^{\mu\nu} - p^\mu p^\nu) \tag{33}$$

Condition (32) requires

$$\lim_{p^2=0} p^2\Pi(p^2, M) = 0, \tag{34}$$

which means a mild behavior of Π in zero.

3 The non-abelian case

We consider the internal group $SU(2)$ for sake of simplicity. We choose the generators of the Lie algebra $su(2)$ to be $T^a = \frac{1}{2}\tau^a$, with τ^a the Pauli matrices. Then $[T^a, T^b] = i\epsilon^{abc}T^c$. In the non-abelian case the Ward identity is replaced by the Slavnov-Taylor identity and the rôle of the equation of motion of the Stückelberg field is taken up by the equation of motion of the ghost field. At tree-level the action of Yang-Mills theory with a Stückelberg mass is

$$\begin{aligned} S \;\; = \;\; & \int d^4x \Big(-\frac{1}{4}G^a_{\mu\nu}G^{a\mu\nu} + \frac{m^2}{2}\Big(A^a_\mu + F^a_\mu\Big)^2 \\ & \qquad + B^a\partial A^a - \bar{c}^a\partial_\mu D^\mu[A]c^a\Big), \end{aligned} \tag{35}$$

where the field strength $G^a_{\mu\nu}$ is given by

$$G^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + \epsilon^{abc}A^b_\mu A^c_\nu \tag{36}$$

and the pure gauge vector field F_μ is

$$F_\mu = F^a_\mu T^a = \frac{i}{2}\Omega\partial_\mu\Omega^\dagger\,. \tag{37}$$

Ω is a unitary matrix parameterized by the fields ϕ^a:

$$m\Omega = \phi_0 \mathbf{1} + i\vec{\phi}\cdot\vec{\tau} \tag{38}$$

with $\phi_0 = \sqrt{m^2 - \phi_a^2}$.

The Slavnov-Taylor identities are derived from the BRST [23] invariance of the action

$$\begin{array}{lll} s\ A_{a\mu} = D[A]_{ab\mu} c_b, & s\ \bar{c}_a = B_a, & s\ \phi_a = \frac{1}{2}\left(c_a \phi_0 - \epsilon_{abc} c_b \phi_c\right), \\ s\ c_a = -\frac{1}{2}\epsilon_{abc} c_b c_c, & s\ B_a = 0, & s\ \phi_0 = -\frac{1}{2} c_a \phi_a \end{array} \tag{39}$$

i.e. (matter will be omitted, group indices and integration over space-time are understood)

$$-\ \mathcal{S}(W) = J^\mu_A W_{A^{*\mu}} + J_\phi W_{\phi^*} + J_{\bar{c}} W_B + J_c W_{c^*} = 0 \tag{40}$$

$$\mathcal{S}(\Gamma) = \Gamma_{A_\mu}\Gamma_{A^{*\mu}} + \Gamma_\phi \Gamma_{\phi^*} + B\Gamma_{\bar{c}} + \Gamma_c \Gamma_{c^*} = 0. \tag{41}$$

A star superscript over a field variable denotes the corresponding antifield [27, 28]. The relevant local cohomology of the linearized classical Slavnov-Taylor operator has been studied in [29].

The $B-$field equation of motion is

$$J_B + \partial^\mu W_{A^\mu} = 0 \tag{42}$$

$$\Gamma_B = \partial^\mu A_\mu. \tag{43}$$

The ghost equation of motion is

$$-J_{\bar{c}} + \partial^\mu W_{A^{*\mu}} = 0 \tag{44}$$

$$\Gamma_{\bar{c}} = -\partial^\mu \Gamma_{A^{*\mu}}. \tag{45}$$

From the eqs. (40-45) one gets

$$\begin{array}{l} W_{A^\mu B} = -i\dfrac{p_\mu}{p^2}, \quad W_{A^\mu \phi} = 0, \quad W_{BB} = 0, \\ W_{B\phi} = W_{\phi^* \bar{c}}, \quad W_{A^{*\mu}\bar{c}} = i\dfrac{p_\mu}{p^2} \end{array} \tag{46}$$

and similarly

$$\begin{array}{l} \Gamma_{BA^\mu} = -ip_\mu, \quad \Gamma_{B\phi} = 0, \quad \Gamma_{BB} = 0, \\ \Gamma_{\bar{c}c} = -ip^\mu \Gamma_{A^{*\mu}c}, \quad \Gamma_{A_\mu \phi}\Gamma_{A^{*\mu}c} + \Gamma_{\phi\phi}\Gamma_{\phi^* c} = 0. \end{array} \tag{47}$$

Eq. (47) implies

$$W_{A^{*\mu}\bar{c}} = \Gamma_{A^{*\mu}c} W_{c\bar{c}} = \Gamma_{A^{*\mu}c} \Gamma_{c\bar{c}}^{-1} = i \frac{p_\mu}{p^2}. \tag{48}$$

Now we use the above equations to get the longitudinal part of the two-point vertex function. We consider the relevant components of the matrix product

$$W\Gamma = -1. \tag{49}$$

We get

$$W_{A^\mu A^\rho}\Gamma_{A_\rho A^\nu} + W_{A^\mu B}\Gamma_{BA^\nu} = -g_{\mu\nu} \quad \Longrightarrow W_T\Gamma_T = -1 \tag{50}$$
$$W_{A^\mu A^\rho}\Gamma_{A_\rho B} + W_{A^\mu \phi}\Gamma_{\phi B} = 0 \tag{51}$$
$$W_{A^\mu A^\rho}\Gamma_{A_\rho \phi} + W_{A^\mu \phi}\Gamma_{\phi\phi} = 0 \tag{52}$$
$$W_{BA^\rho}\Gamma_{A_\rho A^\mu} + W_{B\phi}\Gamma_{\phi A^\mu} = 0 \quad \Longrightarrow i\Gamma_L \frac{p_\mu}{p^2} + W_{B\phi}\Gamma_{\phi A^\mu} = 0 \tag{53}$$
$$W_{BA^\rho}\Gamma_{A_\rho B} + W_{B\phi}\Gamma_{\phi B} = -1 \tag{54}$$
$$W_{BA_\mu}\Gamma_{A^\mu\phi} + W_{B\phi}\Gamma_{\phi\phi} = 0 \tag{55}$$
$$W_{\phi B}\Gamma_{BA^\mu} + W_{\phi\phi}\Gamma_{\phi A^\mu} = 0 \quad \Longrightarrow -ip_\mu W_{B\phi} + W_{\phi\phi}\Gamma_{\phi A^\mu} = 0 \tag{56}$$
$$W_{\phi A^\rho}\Gamma_{A_\rho B} + W_{\phi B}\Gamma_{BB} + W_{\phi\phi}\Gamma_{\phi B} = 0 \tag{57}$$
$$W_{\phi A^\rho}\Gamma_{A_\rho \phi} + W_{\phi B}\Gamma_{B\phi} + W_{\phi\phi}\Gamma_{\phi\phi} = -1 \quad \Longrightarrow W_{\phi\phi}\Gamma_{\phi\phi} = -1 \tag{58}$$

From eqs. (53) and (56) we get

$$\Gamma_L = -\frac{p^2 W_{B\phi}^2}{W_{\phi\phi}}. \tag{59}$$

The requirement that Γ has no poles in $p^2 = 0$ gives

$$\lim_{p^2=0} (\Gamma_L - \Gamma_T) = 0. \tag{60}$$

Then from eqs. (50), (59) and (60)

$$\lim_{p^2=0} \left(W_T - \frac{W_{\phi\phi}}{p^2 W_{B\phi}^2} \right) = 0. \tag{61}$$

3.1 Construction of the unphysical modes

As in the previous section we construct the fields that describe conveniently the unphysical modes

$$X = -\left.\frac{W_{\phi\phi}}{2p^2W_{B\phi}^2}\right|_{p^2=0} B + \left.\frac{1}{p^2W_{B\phi}}\right|_{p^2=0}\phi = -\frac{1}{2}W_T(0)B + \left.\frac{1}{p^2W_{B\phi}}\right|_{p^2=0}\phi. \quad (62)$$

The existence of a linear combination of fields that has no pole at $p^2 = 0$

$$Y^\mu = aA^\mu + b\partial^\mu B + c\partial^\mu X \tag{63}$$

requires

$$\begin{aligned} W_{Y^\mu A^\nu} &= aW_{A^\mu A^\nu} + b\frac{p_\mu p_\nu}{p^2} - c\frac{p_\mu p_\nu}{p^2}\frac{W_T(0)}{2} = 0 \\ W_{Y^\mu B} &= -ia\frac{p_\mu}{p^2} - ic\frac{p_\mu}{p^2} = 0 \\ W_{Y^\mu X} &= ia\frac{p_\mu}{p^2}\frac{W_T(0)}{2} - ib\frac{p_\mu}{p^2} = 0. \end{aligned} \tag{64}$$

This set of equations has a non-trivial solution only if

$$W_T(0) = \lim_{p^2=0}\frac{W_{\phi\phi}}{p^2W_{B\phi}^2} \tag{65}$$

which is guaranteed by the requirement in eq. (61) and it is the condition necessary in order that the determinant of the residuum

$$\mathcal{R}_{ij}(p) = \lim_{p^2=0} p^2W_{ij}(p) \tag{66}$$

is zero. Thus

$$\begin{aligned} Y^\mu &= a\left(A^\mu + \frac{W_T(0)}{2}\partial^\mu B - \partial^\mu X\right) \\ &= a\left(A^\mu + W_T(0)\partial^\mu B - \left.\frac{1}{p^2W_{B\phi}}\right|_{p^2=0}\partial^\mu\phi\right). \end{aligned} \tag{67}$$

Thus we can choose another doublet of fields. Instead of B, X one can use the scalar part of the fields

$$B^\mu \equiv \frac{1}{W_T(0)}\left(-A^\mu + \left.\frac{1}{p^2W_{B\phi}}\right|_{p^2=0}\partial^\mu\phi\right) \tag{68}$$

$$X^\mu \equiv \frac{1}{2}\left(A^\mu + \left.\frac{1}{p^2W_{B\phi}}\right|_{p^2=0}\partial^\mu\phi\right). \tag{69}$$

4 Diagrammatic approach

The results of the previous sections allow us to trace the identities for a diagrammatic approach to the problem of physical unitarity [24][25] . The unphysical states are expected to be massless modes. Few elements support this statement. In particular the $B-$ equation of motion in (42) states that the connected two-point function of the vector field is pure transverse and therefore it contains a pole at zero mass describing a scalar particle. Consequently if we assume that we have asymptotic states the BRST transformations in eq. (39) require that the fields B, ϕ describe massless modes. We have seen in the previous section that only two boson massless modes exist, with opposite metric. As a last comment before entering deeply in the calculation we notice that due to eq. (46) the $B-$field never appear as an internal line in the actual calculation of Feynman amplitudes. In verifying physical unitarity directly on amplitudes we have to consider the projection operators corresponding to the massless modes described by the fields $A_\mu, \phi, c, \bar{c}$. Due to the eq. (46), there is no mixing between the A_μ and the ϕ fields.

Let us formulate the problem in a schematic way. We have a set of fields

$$\psi_i = \{A_{a\mu}, \phi_a, B_a, c_a, \bar{c}_a, \ldots\} \tag{70}$$

The two-point function can be expanded around the poles

$$W_{ij}(x) = \sum_\lambda \int d^3p \frac{1}{2E_\lambda} \langle 0|\psi_i(x)|\lambda\vec{p}\rangle \rho_\lambda^{-1} \langle \lambda\vec{p}|\psi_j(0)|0\rangle, \tag{71}$$

where the index λ collects all internal indices (the mass is m_λ). The states are normalized by

$$\langle \lambda\vec{p}|\lambda'\vec{p}'\rangle = 2E_p \rho_\lambda \delta_{\lambda\lambda'} \delta_3(p-p') \tag{72}$$

where ρ_λ is the metric tensor. The *wave functions* are introduced

$$f_{\lambda pi}(x) = \langle 0|\psi_i(x)|\lambda\vec{p}\rangle \tag{73}$$

and on them it is convenient to introduce the bilinear form (all asymptotic fields obey Klein-Gordon equations)

$$(g_{\lambda'}, f_\lambda) = i\rho_\lambda \delta_{\lambda\lambda'} \sum_i \int d^3x g^*_{\lambda' i} \overleftrightarrow{\partial}_0 f_{\lambda i}. \tag{74}$$

The wave functions form a vector space $\mathcal{V}$. The form in eq. (74) allows to define a dual space $\widetilde{\mathcal{V}}$. In particular we can define an *orthogonal* function $\tilde{f}_{\lambda p} \in \widetilde{\mathcal{V}}$ for any $f_{\lambda p} \in \mathcal{V}$ such that

$$\left(\tilde{f}_{\lambda' p'}, f_{\lambda p}\right) = \rho_\lambda \delta_{\lambda\lambda'} 2E_p \delta_3(p - p'). \tag{75}$$

This unusual structure is necessary in order to deal with a degenerate scalar product in $\mathcal{V}$. For instance there are wave functions of gradients of a massless scalar field. Moreover there are more fields than massless modes.

Now we are in place to define the S-matrix elements in terms of the connected amplitude by using the operator (for an outgoing mode)

$$\begin{aligned} &\langle \dots, \vec{p}\lambda, \dots |S| \dots \rangle = \\ &\dots \left(-\sum_{ij} \int d^4x f^*_{\lambda pi}(x) d^4y \Gamma_{ij}(x-y) \frac{\delta}{\delta J_j(y)} \right) \dots W[J]\big|_{J=0}. \end{aligned} \tag{76}$$

The above equation hints to introduce the operation of *truncation* on the functional W by

$$W_{\widehat{\psi_i(p)}}[J] = -\lim_{p^2=m_\lambda^2} \int d^4x d^4y \exp(ipx) \Gamma_{ij}(x-y) \frac{\delta}{\delta J_j(y)} W[J] \tag{77}$$

i.e. external leg, starting with index i, is removed and the momentum is taken on-shell. The S-matrix element is recovered by folding in the wave function of the mode

$$\langle \dots, \vec{p}\lambda, \dots |S| \dots \rangle = \dots \sum_i f^*_{\lambda pi}\, W_{\widehat{\psi_i(p)}}[J]\Big|_{J=0}. \tag{78}$$

The more familiar reduction formula (LSZ-formalism) takes the form

$$-i \sum_k \int d^4x \tilde{f}^*_{\lambda pk}(x) \rho_\lambda (\Box + m_\lambda^2) \frac{\delta}{\delta J_k(x)}. \tag{79}$$

4.1 Diagrammatic unitarity

Consider now the functional

$$-\int d^4x \exp(ipx) \Box \frac{\delta}{\delta J_B(x)}\, W[J]\big|_{p^2=0} = \lim_{p^2=0} p^2 W_{Bj}(p) W_{\widehat{\psi_j(p)}}[J] \tag{80}$$

The relevant quantity is then the residuum

$$\mathcal{R}_{ij}(p) = \lim_{p^2=0} p^2 W_{ij}(p) \tag{81}$$

and in particular eq. (80) contains only

$$\begin{aligned}\mathcal{R}_{BA^\mu}(p) &= ip_\mu \\ \mathcal{R}_{B\phi}(p) &= \lim_{p^2=0} p^2 W_{B\phi^*}(p)\end{aligned} \tag{82}$$

as one sees from eq. (46).

According to eq. (76) in order to check the required cancellation among unphysical modes in the unitarity equation

$$\langle f|i\rangle = \sum_n \langle f|S^\dagger|n\rangle\langle n|S|i\rangle \tag{83}$$

we have to evaluate terms as (the dependence of W on the external currents is understood)

$$\begin{aligned}&\int \frac{d^3p}{2p}\left(W^*_{\widehat{A_\mu(p)}}\mathcal{R}_{A^\mu A^\nu}(p)W_{\widehat{A_\nu(p)}} + W^*_{\widehat{\phi(p)}}\mathcal{R}_{\phi\phi}(p)W_{\widehat{\phi(p)}}\right)\\ &= \int \frac{d^3p}{2p}\left(-W^*_{\widehat{A_\mu(p)}}W_T(0)p^\mu p^\nu W_{\widehat{A_\nu(p)}} + W^*_{\widehat{\phi(p)}}\mathcal{R}_{\phi\phi}(p)W_{\widehat{\phi(p)}}\right)\\ &= \int \frac{d^3p}{2p}W_T(0)\left(-W^*_{\widehat{A_\mu(p)}}p^\mu p^\nu W_{\widehat{A_\nu(p)}} + W^*_{\widehat{\phi(p)}}\mathcal{R}^2_{\phi B}W_{\widehat{\phi(p)}}\right)\\ &= \int \frac{d^3p}{2p}\frac{W_T(0)}{2}\Big[\left(ip^\mu W^*_{\widehat{A_\mu(p)}} + W^*_{\widehat{\phi(p)}}\mathcal{R}_{\phi B}\right)\left(ip^\nu W_{\widehat{A_\nu(p)}} + \mathcal{R}_{\phi B}W_{\widehat{\phi(p)}}\right)\\ &+ \left(-ip^\mu W^*_{\widehat{A_\mu(p)}} + W^*_{\widehat{\phi(p)}}\mathcal{R}_{\phi B}\right)\left(-ip^\nu W_{\widehat{A_\nu(p)}} + \mathcal{R}_{\phi B}W_{\widehat{\phi(p)}}\right)\Big]\end{aligned} \tag{84}$$

In deriving eq. (84) we have used eq. (61).

By comparing eqs. (80-82) with eq. (84) we can identify in two terms in round brackets the operation

$$ip^\nu W_{\widehat{A_\nu(p)}} + \mathcal{R}_{\phi B}W_{\widehat{\phi(p)}} = -\int d^4x \exp(ipx)\Box_x\, W_{B(x)}\Big|_{p^2=0}. \tag{85}$$

If the unitarity equation (83) contains more than one particle in the final state n, then we have to consider the reduction formula for a further momentum q. The unphysical states are described by the massless modes in A_μ and ϕ. According to eq. (40) we have

$$W_{B(x)A_\mu(y)} = W_{\bar{c}(x)A^*_\mu(y)} \tag{86}$$

$$W_{B(x)\phi(y)} = W_{\bar{c}(x)\phi^*(y)}. \tag{87}$$

Then from eqs. (85) and (46)

$$ip^\nu W_{\widehat{A_\nu(p)}A_\mu(y)} + \mathcal{R}_{\phi B}(p)W_{\widehat{\phi(p)}A_\mu(y)} = \mathcal{R}_{\bar{c}c}(p)W_{\widehat{\bar{c}(p)}A^*_\mu(y)}, \tag{88}$$

from where we eventually get the residuum of the pole at $q^2 = 0$

$$\begin{aligned}
&-W_T(0)q^\sigma q^\tau \left(ip^\nu W_{\widehat{A_\nu(p)}\widehat{A_\tau(q)}} + \mathcal{R}_{\phi B}(p) W_{\widehat{\phi(p)}\widehat{A_\sigma(q)}} \right) \\
&= \mathcal{R}_{\bar{c}c}(p)\mathcal{R}_{A^*_\sigma \bar{c}}(q) W_{\widehat{\bar{c}(p)}\widehat{c(q)}} \\
&= iq^\sigma \mathcal{R}_{\bar{c}c}(p) W_{\widehat{\bar{c}(p)}\widehat{c(q)}},
\end{aligned} \tag{89}$$

where the last step is due to eq. (46).

Similarly from eq. (87) one gets

$$ip^\nu W_{\widehat{A_\nu(p)}\phi(y)} + \mathcal{R}_{\phi B}(p) W_{\widehat{\phi(p)}\phi(y)} = \mathcal{R}_{\bar{c}c}(p) W_{\widehat{\bar{c}(p)}\phi^*(y)}, \tag{90}$$

where the residuum at $q^2 = 0$ is

$$\begin{aligned}
&\mathcal{R}_{\phi\phi}(q) \left(ip^\nu W_{\widehat{A_\nu(p)}\widehat{\phi(q)}} + \mathcal{R}_{\phi B}(p) W_{\widehat{\phi(p)}\widehat{\phi(q)}} \right) \\
&= \mathcal{R}_{\bar{c}c}(p)\mathcal{R}_{\phi^*\bar{c}}(q) W_{\widehat{\bar{c}(p)}\widehat{c(q)}} = \mathcal{R}_{\bar{c}c}(p)\mathcal{R}_{B\phi}(q) W_{\widehat{\bar{c}(p)}\widehat{c(q)}},
\end{aligned} \tag{91}$$

where the last step is due to eq. (46).

The first term in eq. (84), after the insertion of an extra intermediate state with momentum q reads

$$\begin{aligned}
&\int \frac{d^3p}{2p} \int \frac{d^3q}{2q} \frac{W_T(0)}{2} \\
&\Big[- W_T(0) q^\sigma \left(ip^\mu W^*_{\widehat{A_\mu(p)}\widehat{A_\sigma(q)}} + W^*_{\widehat{\phi(p)}\widehat{A_\sigma(q)}} \mathcal{R}_{\phi B} \right) \\
&q^\tau \left(ip^\nu W_{\widehat{A_\nu(p)}\widehat{A_\tau(q)}} + \mathcal{R}_{\phi B} W_{\widehat{\phi(p)}\widehat{A_\tau(q)}} \right) \\
&+\mathcal{R}_{\phi\phi}(q) \left(ip^\mu W^*_{\widehat{A_\mu(p)}\widehat{\phi(q)}} + W^*_{\widehat{\phi(p)}\widehat{\phi(q)}} \mathcal{R}_{\phi B} \right) \\
&\left(ip^\nu W_{\widehat{A_\nu(p)}\widehat{\phi(q)}} + \mathcal{R}_{\phi B} W_{\widehat{\phi(p)}\widehat{\phi(q)}} \right) \Big]
\end{aligned} \tag{92}$$

By using eqs. (89) and (91) we get

$$\begin{aligned}
&= \int \frac{d^3p}{2p} \int \frac{d^3q}{2q} \frac{W_T(0)}{2} \\
&\Big[q^\sigma \left(ip^\mu W^*_{\widehat{A_\mu(p)}\widehat{A_\sigma(q)}} + W^*_{\widehat{\phi(p)}\widehat{A_\sigma(q)}} \mathcal{R}_{\phi B} \right) i\mathcal{R}_{\bar{c}c}(p) W_{\widehat{\bar{c}(p)}\widehat{c(q)}} \\
&+ \left(ip^\mu W^*_{\widehat{A_\mu(p)}\widehat{\phi(q)}} + W^*_{\widehat{\phi(p)}\widehat{\phi(q)}} \mathcal{R}_{\phi B} \right) \mathcal{R}_{\bar{c}c}(p)\mathcal{R}_{B\phi}(q) W_{\widehat{\bar{c}(p)}\widehat{c(q)}} \Big] \\
&= \int \frac{d^3p}{2p} \int \frac{d^3q}{2q} \frac{W_T(0)}{2} \mathcal{R}_{\bar{c}c}(p) W_{\widehat{\bar{c}(p)}\widehat{c(q)}} \\
&\Big[ip^\mu \left(iq^\sigma W^*_{\widehat{A_\mu(p)}\widehat{A_\sigma(q)}} + \mathcal{R}_{B\phi}(q) W^*_{\widehat{A_\mu(p)}\widehat{\phi(q)}} \right) \\
&+\mathcal{R}_{B\phi}(p) \left(iq^\sigma W^*_{\widehat{\phi(p)}\widehat{A_\sigma(q)}} + \mathcal{R}_{B\phi}(q) W^*_{\widehat{\phi(p)}\widehat{\phi(q)}} \right) \Big]
\end{aligned} \tag{93}$$

Now we use again eqs. (89) and (91) on the line with momentum q

$$
\begin{aligned}
&= \int \frac{d^3p}{2p} \int \frac{d^3q}{2q} \frac{W_T(0)}{2} \mathcal{R}_{\bar{c}c}(p) W_{\widehat{\bar{c}(p)}\widehat{c(q)}} \\
&\Big[\Big(\frac{\mathcal{R}^*_{\bar{c}c}(q)}{W_T(0)} W^*_{\widehat{\bar{c}(q)}\widehat{c(p)}} \Big) + \mathcal{R}_{B\phi}(p) \Big(\mathcal{R}^*_{\bar{c}c}(q) \frac{\mathcal{R}_{B\phi}(q)}{\mathcal{R}_{\phi\phi}(q)} W^*_{\widehat{\bar{c}(q)}\widehat{c(p)}} \Big) \Big] \\
&= \int \frac{d^3p}{2p} \int \frac{d^3q}{2q} \mathcal{R}_{\bar{c}c}(p) \mathcal{R}^*_{\bar{c}c}(q) W^*_{\widehat{\bar{c}(q)}\widehat{c(p)}} W_{\widehat{\bar{c}(p)}\widehat{c(q)}} \qquad (94)
\end{aligned}
$$

which cancels the contribution of the ghost with momentum p and anti-ghost with momentum q.

The second term in eq. (84) gives

$$
= \int \frac{d^3p}{2p} \int \frac{d^3q}{2q} \mathcal{R}^*_{\bar{c}c}(p) \mathcal{R}_{\bar{c}c}(q) W_{\widehat{\bar{c}(q)}\widehat{c(p)}} W^*_{\widehat{\bar{c}(p)}\widehat{c(q)}} \qquad (95)
$$

i.e. a quantity that cancels the contribution of the ghost with momentum q and anti-ghost with momentum p.

5 Operatorial approach

Physical unitarity can also be analyzed by using a complementary operatorial approach [14, 15, 16]. We start by defining the S-matrix elements from the connected generating functional by means of the following equation

$$
S = \, : \Sigma : W|_{J=\psi^*=0} \qquad (96)
$$

ψ^* denotes collectively the antifields coupled to the fields ψ in eq.(70), J the sources coupled to ψ. The normal product prescription is indicated by vertical dots.

In the bosonic massless sector it is convenient to use the (overcomplete) set of states spanned by ϕ, B, A_μ. Then the operator Σ takes the form (suppressing the color indices)

$$
\begin{aligned}
\Sigma \;\; = \;\; \exp\Big(\int & d^4p \, (A^\mu \; \Gamma_{\mu\nu} \frac{\delta}{\delta J_\nu} + A^\mu \; \Gamma_{\mu\phi} \frac{\delta}{\delta J_\phi} + A^\mu \; \Gamma_{\mu B} \frac{\delta}{\delta J_B} \\
& + B \; \Gamma_{B\nu} \frac{\delta}{\delta J_\nu} + B \; \Gamma_{B\phi} \frac{\delta}{\delta J_\phi} \\
& + \phi \; \Gamma_{\phi\nu} \frac{\delta}{\delta J_\nu} + \phi \; \Gamma_{\phi\phi} \frac{\delta}{\delta J_\phi} + \phi \; \Gamma_{\phi B} \frac{\delta}{\delta J_B} \\
& + c \; \Gamma_{c\bar{c}} \frac{\delta}{\delta J_{\bar{c}}} + \bar{c} \; \Gamma_{\bar{c}c} \frac{\delta}{\delta J_c}) \Big) \qquad (97)
\end{aligned}
$$

where use has been made of eq.(47).

The relevant asymptotic BRST charge Q associated with the BRST operator in eq.(39) is given by

$$[Q, A_\mu] = \Gamma_{cA^*_\mu} c\,, \qquad [Q, \phi] = \Gamma_{c\phi^*} c\,,$$
$$[Q, c]_+ = 0\,, \qquad [Q, \bar{c}]_+ = B\,, \qquad [Q, B] = 0\,. \tag{98}$$

The commutator between Q and the operator $:\Sigma:$ yields

$$\begin{aligned}[Q, :\Sigma:] \;\; = \;\; &: \int d^4p \Big[\; c \, \Big((\Gamma_{cA^*_\mu} \Gamma_{\mu\nu} + \Gamma_{c\phi^*}\Gamma_{\phi\nu}) \frac{\delta}{\delta J_\nu} \\ &+ (\Gamma_{cA^*_\mu} \Gamma_{\mu\phi} + \Gamma_{c\phi^*}\Gamma_{\phi\phi}) \frac{\delta}{\delta J_\phi} \\ &+ (\Gamma_{cA^*_\mu} \Gamma_{\mu B} + \Gamma_{c\phi^*}\Gamma_{\phi B}) \frac{\delta}{\delta J_B} \Big) \\ &+ B \; \Gamma_{\bar{c}c} \frac{\delta}{\delta J_c} \Big] \Sigma : \end{aligned} \tag{99}$$

We now use the following three relations obtained by differentiating the STI in eq.(41) w.r.t. to the ghost c and A_ν, ϕ, B respectively:

$$\Gamma_{cA^*_\mu} \Gamma_{\mu\nu} + \Gamma_{c\phi^*}\Gamma_{\nu\phi} = 0\,, \tag{100}$$
$$\Gamma_{cA^*_\mu} \Gamma_{\mu\phi} + \Gamma_{c\phi^*}\Gamma_{\phi\phi} = 0\,, \tag{101}$$
$$\Gamma_{cA^*_\mu} \Gamma_{\mu B} + \Gamma_{c\phi^*}\Gamma_{\phi B} + \Gamma_{c\bar{c}} = 0\,. \tag{102}$$

Then eq.(99) simplifies to

$$[Q, :\Sigma:] =: \int d^4p \Big[- c \; \Gamma_{c\bar{c}} \frac{\delta}{\delta J_B} + B \; \Gamma_{\bar{c}c} \frac{\delta}{\delta J_c} \Big] \Sigma : \,. \tag{103}$$

The above expression equals the commutator $[:\Sigma:, \mathcal{S}]$ (one again needs eqs.(100)-(102)):

$$[Q, :\Sigma:] = [:\Sigma:, \mathcal{S}]\,. \tag{104}$$

From the above equation it follows that Q is a conserved charge:

$$[Q, S] = [Q, :\Sigma: W|_{J=\psi^*=0}] = 0\,. \tag{105}$$

Moreover from eq.(98) Q is nilpotent.

Now we are in a position to characterize the Hilbert space $\mathcal{H}_{\rm phys} = {\rm Ker}\; Q/{\rm Im}\; Q$. From eq.(98) the three massive states of the gauge field A_μ

belong to $\mathcal{H}_{\text{phys}}$. The massless unphysical states can be analyzed as follows. B is in the kernel of Q but is Q-exact, as it follows from eq.(98). Moreover the state X in eq.(62) does not belong to the kernel of Q. Finally Y_μ in eq.(67) is in the kernel of Q, since

$$
\begin{aligned}
[Q, Y_\mu] &= a\Big(\Gamma_{A^*_\mu c} - \frac{1}{p^2}\frac{1}{W_{\phi B}} ip_\mu \Gamma_{\phi^* c}\Big)c \\
&= \frac{a}{W_{\phi B}}\Big(W_{\phi B}\Gamma_{A^*_\mu c} - \frac{ip_\mu}{p^2}\Gamma_{\phi^* c}\Big)c \\
&= \frac{a}{p^2 W_{\phi B}}\frac{(-ip_\mu)}{\Gamma_{\phi\phi}}\Big(\Gamma_{A_\nu \phi(-p)}\Gamma_{A^*_\nu(-p)c(p)} + \Gamma_{\phi\phi(-p)}\Gamma_{\phi^*(-p)c(p)}\Big)c \\
&= 0
\end{aligned}
\tag{106}
$$

by virtue of eq.(101). However such a linear combination does not have a pole at $p^2 = 0$, as it follows from the analysis of Sect. 3. $\bar{c}$ is not in the kernel of Q, while c forms a BRST doublet together with ϕ (see eq.(98)).

We conclude that the only physical states are given by the three transverse massive polarizations of the gauge fields A_μ.

6 Conclusions

Under the assumption that the theory can be defined in some subtraction scheme fulfilling the ST identities, the ghost equation and the B-equation, in the Landau gauge the unphysical pole of the vector meson propagator is a single pole located at $p^2 = 0$. Being a consequence of the symmetries of the theory, this result holds irrespective of the intricacies of the subtraction operation to be envisaged in the context of a non power-counting renormalizable theory. We notice that if additional singularities (beyond the single pole of the physical massive states) are generated in W_T by the subtraction scheme, they influence the (asymptotic) 2-point correlation function of $F^a_{\mu\nu}$ and therefore affect the physical observables of the theory.

A detailed study of physical unitarity has been carried out within this framework. We find that in the Landau gauge physical unitarity is fulfilled provided that the normalization condition in eq.(6) is imposed. This is an all-order universal constraint on candidate subtraction schemes (like dimensional regularization [26]) required in order to achieve a consistent quantum definition of the Stückelberg model.

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A Other covariant gauges

We briefly consider also the case of general covariant gauges given by the gauge fixing term

$$S_{g.f.} = \int d^4x \; 2\,Tr\left(\frac{\alpha}{2}B^2 + B\partial^\mu A_\mu - \bar{c}\partial^\mu D[A]_\mu c\right). \tag{107}$$

The Slavnov-Taylor identities (40) and (41) as well as the ghost equations (44) and (44) unchanged. The B-field equation is now

$$\alpha W_B + J_B + \partial^\mu W_{A^\mu} = 0 \tag{108}$$

$$\Gamma_B = \alpha B + \partial^\mu A_\mu. \tag{109}$$

Consequently eqs. (46) become

$$\begin{aligned} &W_{A^\mu B} = -i\frac{p_\mu}{p^2}, \quad W_{A^\mu\phi} = -i\alpha\,\frac{p_\mu}{p^2}W_{B\phi}, \quad W_{BB} = 0, \\ &W_{B\phi} = W_{\phi^*\bar{c}}, \quad W_{A^{*\mu}\bar{c}} = i\frac{p_\mu}{p^2} \end{aligned} \tag{110}$$

and similarly

$$\begin{aligned} &\Gamma_{BA^\mu} = -ip_\mu, \quad \Gamma_{B\phi} = 0, \quad \Gamma_{BB} = \alpha, \\ &\Gamma_{\bar{c}c} = -ip^\mu\Gamma_{A^{*\mu}c}, \quad \Gamma_{A_\mu\phi}\Gamma_{A^{*\mu}c} + \Gamma_{\phi\phi}\Gamma_{\phi^*c} = 0. \end{aligned} \tag{111}$$

We impose the relation between the two-point functions W and Γ as in eq. (49) and thus get some conditions as in eqs. (50 -59).

$$\begin{aligned} &W_{A^\mu A^\rho}\Gamma_{A_\rho A^\nu} + W_{A^\mu B}\Gamma_{BA^\nu} + W_{A^\mu\phi}\Gamma_{\phi A^\nu} = -g_{\mu\nu} \\ &\Longrightarrow W_T\Gamma_T = -1, \quad W_L\Gamma_L - i\alpha W_{B\phi}\frac{1}{p^2}p^\nu\Gamma_{\phi A^\nu} = 0 \end{aligned} \tag{112}$$

$$\begin{aligned} &W_{A^\mu A^\rho}\Gamma_{A_\rho B} + W_{A^\mu B}\Gamma_{BB} = 0 \\ &\Longrightarrow W_L - \alpha\frac{1}{p^2} = 0 \end{aligned} \tag{113}$$

$$
\begin{aligned}
&W_{A^\mu A^\rho}\Gamma_{A_\rho\phi} + W_{A^\mu\phi}\Gamma_{\phi\phi} = 0 \\
&\Longrightarrow W_L p^\rho \Gamma_{A^\rho\phi} - i\alpha W_{B\phi}\Gamma_{\phi\phi} = 0 \qquad (114) \\
&W_{BA^\rho}\Gamma_{A_\rho A^\mu} + W_{B\phi}\Gamma_{\phi A^\mu} = 0 \\
&\Longrightarrow i\Gamma_L + W_{B\phi}p_\mu\Gamma_{\phi A^\mu} = 0 \qquad (115) \\
&W_{BA^\rho}\Gamma_{A_\rho B} + W_{B\phi}\Gamma_{\phi B} = -1 \qquad (116) \\
&W_{BA_\mu}\Gamma_{A^\mu\phi} + W_{B\phi}\Gamma_{\phi\phi} = 0 \\
&\Longrightarrow i\frac{p_\mu}{p^2}\Gamma_{A^\mu\phi} + W_{B\phi}\Gamma_{\phi\phi} = 0 \qquad (117) \\
&W_{\phi A_\nu}\Gamma_{A^\nu A^\mu} + W_{\phi B}\Gamma_{BA^\mu} + W_{\phi\phi}\Gamma_{\phi A^\mu} = 0 \\
&\Longrightarrow i\alpha W_{B\phi}\Gamma_L - iW_{B\phi}p^2 + W_{\phi\phi}p_\mu\Gamma_{\phi A^\mu} = 0 \qquad (118) \\
&W_{\phi A^\rho}\Gamma_{A_\rho B} + W_{\phi B}\Gamma_{BB} + W_{\phi\phi}\Gamma_{\phi B} = 0 \qquad (119) \\
&W_{\phi A^\rho}\Gamma_{A_\rho\phi} + W_{\phi B}\Gamma_{B\phi} + W_{\phi\phi}\Gamma_{\phi\phi} = -1 \\
&\Longrightarrow i\alpha\frac{p^\rho}{p^2}W_{B\phi}\Gamma_{A_\rho\phi} + W_{\phi\phi}\Gamma_{\phi\phi} = -1. \qquad (120)
\end{aligned}
$$

From the eqs. (112-120) we get

$$W_T = -\frac{1}{\Gamma_T} \qquad (121)$$

$$W_L = \frac{\alpha}{p^2} \qquad (122)$$

$$\Gamma_L = p^2\Gamma_{\phi\phi}W_{B\phi}^2 \qquad (123)$$

$$\Gamma_{\phi A_\rho} = -ip_\rho\Gamma_{\phi\phi}W_{B\phi} \qquad (124)$$

$$W_{\phi\phi}\Gamma_{\phi\phi} = -1 + \frac{\alpha}{p^2}\Gamma_L. \qquad (125)$$

Eq. (122) shows that the connected two-point function $W_{A_\mu A_\nu}$ potentially develops a double pole at $p^2 = 0$.

From eq.(125), (123) and eq.(124) we get

$$
\begin{aligned}
W_{\phi\phi} &= \frac{1}{\Gamma_{\phi\phi}}\Big(-1 + \frac{\alpha}{p^2}\Gamma_L\Big) \\
&= \frac{1}{\Gamma_{\phi\phi}}\Big(-1 + \alpha\Gamma_{\phi\phi}W_{B\phi}^2\Big) \\
&= \frac{1}{\Gamma_{\phi\phi}}\Big(-1 - \frac{\alpha}{p^2\Gamma_{\phi\phi}}(p^\rho\Gamma_{\phi A_\rho})^2\Big)\,. \qquad (126)
\end{aligned}
$$

We define

$$\Gamma_{\phi A_\mu} = ip_\mu f(p^2)\,, \qquad \Gamma_{A_\mu^*\phi} = ip_\mu g(p^2)\,. \qquad (127)$$

Eq.(126) becomes

$$W_{\phi\phi} = \frac{1}{\Gamma_{\phi\phi}}\Big(-1 + \frac{\alpha}{\Gamma_{\phi\phi}} f^2(p^2)\Big) = \frac{\alpha f^2(p^2) - \Gamma_{\phi\phi}}{(\Gamma_{\phi\phi})^2}\,. \tag{128}$$

Then from eq.(111)

$$\Gamma_{\phi\phi} = -\frac{\Gamma_{\phi A_\mu}\Gamma_{A^*_\mu c}}{\Gamma_{\phi^* c}} = -\frac{p^2 f(p^2) g(p^2)}{\Gamma_{\phi^* c}}\,. \tag{129}$$

By assuming that $\Gamma_{\phi^* c}$ tends to a constant different from zero for $p^2 \to 0$ $\Gamma_{\phi\phi}$ has a zero at $p^2 = 0$. At tree-level it is the only zero of $\Gamma_{\phi\phi}$. Provided that this is true also at the quantum level, we see from eq.(128) that $W_{\phi\phi}$ also potentially (i.e. by excluding that $f(p^2)$ has a zero in $p^2 = 0$) develops a double pole at $p^2 = 0$.

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